Progress Repoprt on Hyper-operations (Tetration)

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This progress report has been prepared as a follow-up to a note submitted by the Authors to the NKS Forum, posted in the News and Announcement section on 3rd October 2004, with reference to an article prepared by the same Authors and published on the Internet on 15-09-2004, with the title: ***Ackermann's Function and New Arithmetical Operation,*** paper mentioned in the NKS Bibliography. For the understanding of the present report, it is advisable to have read that article.

The present report shows some progress in the research so far, particularly concerning the following issues, put together for facilitating any possible co-operation from Researchers interested in this innovative field:

- The definition of rank *s* hyper-operations and of their two inverse operations, together with a formula showing their recursive properties.
- The properties of the infinite hyper-root with application to the super-root, the first inverse hyper-operation of rank 4.
- The relationship between the square super-root and the infinite tower operations.
- A property of the super-logarithm, second inverse hyper-operation of rank 4.
- A recursive hyper-logarithm formula.
- Connections between the square super-root with the Lambert's Function (product-log).
- Properties of the infinite tower operation and its expression by using the product-log function.
- The tetrational notation of numbers by using the tower extension.
- The Numbers' Notation Hyper-format (RRH ©).
- Some tower plots, with integer super-exponents and their properties.
- Possible approximations usable for an acceptable definition of the tetrational function, to the base *e*, as well as of its inverse operation, the natural super-logarithm.

Examples of Terminology and Symbols

$\boxed{s}$	- general hyper-operator of rank *s*, with $\boxed{1} = +$, $\boxed{2} = \times$, $\boxed{3} = \wedge$ and $\boxed{0} = \circ$ (zeration), $\boxed{4} = \#$ (tetration);
$y = {}^{z}x$	- tetration; $x \# z$; *x*-tetrated-*z* or *x*-tower-*z*, i.e. *x* raised to *x*, *z* times;
$y = x \circ z$	- zeration; *x*-zerated-*z* or *x*-ball-*z*; max(x,z)+1 if x≠z, x+2=z+2 if x = z;
$x = y \Delta x$	- deltation; *y*-delta-*x* (inverse function of $y = x \circ z$);
${}^{z}\lceil y \rceil_{s}$	- the *z*-th hyper-root of *y*, rank *s*; for s = 4 : ${}^{z}\lceil y \rceil_{4} = {}^{z}\!\!\sqrt{y}$ (the *z*-th super-root of *y*) and, for s = 3: ${}^{z}\lceil y \rceil_{3} = \sqrt[z]{y}$ (the *z*-th root of *y*);
${}_{x}\lfloor y \rfloor_{s}$	- the hyper-logarithm, base *x*, of *y*, rank *s*, that can also be written as: $\mathrm{hlog}_{s\,x}\, y$; for $s = 4$: ${}_{x}\lfloor y \rfloor_{4} = \mathrm{slog}_x\, y$ (super-log, base *x*, of *y*) and for $s = 3$: ${}_{x}\lfloor y \rfloor_{3} = \log_x y$;
$\mathrm{plog}(x)$	- the ProductLog (Lambert's Function), the inverse function *y* of $x = y \cdot e^{y}$;
$\ln y$	- the natural logarithm of *y*, i.e. : $\log_e y$;
$\mathrm{s}\ln y$	- the natural super-logarithm of *y*, i.e.: $\mathrm{slog}_e\, y$;

Hyper-operations (Tetration)

1 - Hyper-operations of Rank "*s*" and their Inverse Operations

A general hyper-operation of rank "*s*" can be written as follows:

(1) $$\boxed{z = x\boxed{s}y}$$

where x and y are the operands and $\boxed{s}$ stands for the infixed hyper-operator of rank s. As mentioned in [1], paragraph 4.4, *zeration*, *addition*, *multiplication*, *exponentiation*, *tetration* (etc.) are generalized "natural" hyper-operations defined as follows:

(2)

s = 0	*zeration*	$z = x\boxed{0}y = x \circ y$
s = 1	*addition*:	$z = x\boxed{1}y = x + y$
s = 2	*multiplication*:	$z = x\boxed{2}y = x \cdot y = xy$
s = 3	*exponentiation*:	$z = x\boxed{3}y = x \wedge y = x^y$
s = 4	*tetration*:	$z = x\boxed{4}y = x \# y = {}^{y}x$

Any hyper-operation determines two inverse (left and right) operations, which are:

(3) $$\boxed{x = \overset{y\ \ s}{\lceil}z}$$ the <u>*y*-th hyper-root</u> (rank s) of z

(4) $$\boxed{y = \underset{x\ \ s}{\lfloor}z}$$ the <u>hyper-logarithm, base *x*</u>, (rank s) of z

If the direct operation is commutative (s = 0, 1, 2) the two inverse operations coincide, if the direct operation is not commutative (s ≥ 3) , the two inverse operations are different. Moreover, for the direct operations, which belong to the class of *recursive functions*, the following properties are verified:

(5)
$$x + (y+1) = x \circ (x+y) = x + y + 1 \qquad \text{with : } y \neq 0$$
$$x \cdot (y+1) = x + (x \cdot y) = x + xy$$
$$x \wedge (y+1) = x \cdot (x \wedge y) = x \cdot x^y$$
$$x \# (y+1) = x \wedge (x \# y) = x \wedge ({}^{y}x)$$

Therefore, in general, we have the following recursive property:

(6) $$\boxed{x\boxed{s}(y+1) = x\boxed{s-1}\left(x\boxed{s}y\right)}$$

We wish to show here a table of the inverse operations of the hyper-operations of rank s (0 ≤ s ≤ 4) of the root and of the logarithm type [1]. The table shows that the inverse operations for the ranks 3 and 4 are different:

(7)

			Root type	Log type
s = 0	*zeration*	$z = x \circ y$	$x = z\Delta y$	$y = z\Delta x$
s = 1	*addition*:	$z = x + y$	$x = z - y$	$y = z - x$
s = 2	*multiplication*:	$z = xy$	$x = z / y$	$y = z / x$
s = 3	*exponentiation*:	$z = x^y$	$x = \sqrt[y]{z}$	$y = \log_x z$
s = 4	*tetration*:	$z = {}^{y}x$	$x = {}^{y}\!\!\left.\right) z$	$y = \mathrm{slog}_x z$

[1] The two inverse operations of tetration are called <u>super-root</u> and <u>super-logarithm</u> (superlog), respectively.

2 - The Infinite Hyper-root

Concerning the first inverse operation (of the hyper-root type), we also have:

(8)
$$\boxed{\lim_{y\to\infty} {}^{y}\overset{s}{\lceil} z = {}^{\infty}\overset{s}{\lceil} z = {}^{z}\overset{s-1}{\lceil} z}$$

This formula can be implemented, for the first four (s >0) hyper-operations' ranks, as follows:

	s = 1	*addition*:	${}^{\infty}\overset{1}{\lceil} z = {}^{z}\overset{0}{\lceil} z$	$\leftrightarrow$	$z - \infty = z \Delta z = -\infty$
(9)	s = 2	*multiplication*:	${}^{\infty}\overset{2}{\lceil} z = {}^{z}\overset{1}{\lceil} z$	$\leftrightarrow$	$z / \infty = z - z = 0$
	s = 3	*exponentiation*:	${}^{\infty}\overset{3}{\lceil} z = {}^{z}\overset{2}{\lceil} z$	$\leftrightarrow$	$\sqrt[\infty]{z} = z / z = 1$
	s = 4	*tetration*:	${}^{\infty}\overset{4}{\lceil} z = {}^{z}\overset{3}{\lceil} z$	$\leftrightarrow$	${}^{\infty}\big)\overline{z} = \sqrt[z]{z}$

The validity of these operations' invariant formulas, for the hyper-operations' ranks 2 and 3, is easily checked. The validity for s = 1 is a consequence of the definition of zeration and of its inverse operation (deltation). We shall try here to recall a simple demonstration for s = 4. In particular, we wish to show that we can express the "infinite super-root" of a number x (by using symbol $\big)\overline{} = \overset{4}{\lceil}$, for indicating the *super*-root [2], or the rank-4 hyper-root), as follows:

(10)
$$\boxed{{}^{\infty}\big)\overline{x} = \sqrt[x]{x}}$$

In fact, we can write the "infinite tower" (infinite tetration) of a number y in the following way:

$$x = \lim_{n\to\infty} {}^{n}y = {}^{\infty}y$$

In this case, y, the "infinite super-root" of a number x, can be written like this:

$$y = \lim_{n\to\infty} {}^{n}\big)\overline{x} = {}^{\infty}\big)\overline{x}$$

Then, from reference [1], paragraph 6.4, by taking the natural logarithm of ${}^{\infty}y$, we have:

(11) $$\ln {}^{\infty}y = {}^{\infty}y \cdot \ln y$$

so, we have: $$\ln y = (\ln {}^{\infty}y) / {}^{\infty}y = \ln\left[\left({}^{\infty}y\right)^{1/{}^{\infty}y}\right]$$

and, finally: $$y = {}^{\infty}\big)\overline{x} = \left({}^{\infty}y\right)^{1/{}^{\infty}y} = x^{1/x} = \sqrt[x]{x}$$ **QED**[3]

[2] Prefix super- is reserved to rank-4 hyper-operation (tetration). Prefix hyper- is generally used for all the hierarchical ranks.

[3] QED: *Quod Erat Demonstrandum*.

3 – The Square Super-root

We wish to show that we can express the square super-root of a number *x* (with symbol $\overline{)\ }$ or ${}^{2}\overline{)\ }$ to indicate the square super-root operator) as follows:

(12) $$\boxed{{}^{2}\overline{)x} = 1/\,{}^{\infty}\left(1/x\right) = \frac{1}{\left(1/x\right)\#\infty}}$$

In fact, let us take into consideration expression (10) of the infinite super-root of a number *v*:

$${}^{\infty}\overline{)v} = \sqrt[v]{v}$$

by putting $v \to 1/z$, we get: $\quad {}^{\infty}\overline{)1/z} = 1/z^{z} = 1/\,{}^{2}z$

therefore we have: $\quad {}^{2}z = 1/\,{}^{\infty}\overline{)1/z}$

i.e.: $\quad 1/z = {}^{\infty}\left(1/\,{}^{2}z\right)$

and, by putting now: $\quad z = {}^{2}\overline{)x} \qquad$ square super-root of *x* (${}^{2}z = x$)

we obtain: $\quad 1/\,{}^{2}\overline{)x} = {}^{\infty}\left(1/x\right)$

that we can also write [4] as: $\quad :({}^{2}\overline{)x}) = {}^{\infty}\left(:x\right) \qquad$ **QED**

4 – The Super-logarithm

Let us take into consideration the following expressions, directly coming from the definition of the tetration operation and also mentioned in formulas (5) and (6):

(13) $$b\,\hat{}\,({}^{x}b) = {}^{x+1}b$$

which means: $\quad {}^{x}b = \log_b({}^{x+1}b)$

Therefore, by applying the super-logarithm operation, base *b*, we get:

$$x = \text{slog}_b(\log_b({}^{x+1}b))$$

and if we put: $\quad y = {}^{x+1}b$

i.e.: $\quad x + 1 = \text{slog}_b\, y$

which means:

(14) $$\boxed{\text{slog}_b\, y = \text{slog}_b(\log_b y) + 1}$$

and, for $b = e$ we have, for the natural super-logarithm:

(15) $$\text{sln}\, y = \text{sln}(\ln y) + 1$$

[4] In the mentioned formula, we adopt the innovative use of the absolute ":" symbol, with the meaning of "1/…" (the reciprocal of), similarly to the use of the absolute "-" symbol, indicating "0 - …" (the opposite of).

5 – A Recursive Hyper-logarithm Formula

Let us recall definition (1) of a general direct hyper-operation of rank *s*, as well as that of its second inverse operation, the hyper-logarithm:

$$z = x\boxed{s}y \qquad ; \qquad y = \mathop{\lfloor}_{x\ s} z \ .$$

Formula (14) stipulates that, for s = 4 (tetration), we must have:

(16) $$\mathop{\lfloor}_{x\ 4} z = \operatorname{slog}_x z = \operatorname{slog}_x(\log_x z) + 1 \qquad \text{with: } x > 1$$

Formula (16) has a similar expression at level s = 3 (exponentiation) that can be written as follows:

(17) $$\mathop{\lfloor}_{x\ 3} z = \log_x z = \log_x(z/x) + 1 \qquad \text{with: } x > 1$$

The corresponding formulas, for s = 2 and s = 1 are:

(18) $$\mathop{\lfloor}_{x\ 2} z = z/x = \frac{z-x}{x} + 1 \qquad \text{with: } x \neq 0$$

$$\mathop{\lfloor}_{x\ 1} z = z - x = (z\Delta x) - x + 1 \qquad \text{with: } z > x + 2$$

In conclusion, we may presume that the following recursive expression is valid for all the hyper-operations' hierarchy:

(19) $$\boxed{\mathop{\lfloor}_{x\ s} z = \mathop{\lfloor}_{x\ s}(\mathop{\lfloor}_{x\ s-1} z) + 1}$$

or, more traditionally, by indicating with "$\underset{s}{\operatorname{hlog}_x} z$" the (rank *s*)-hyper-logarithm, base *x*, of number *z*:

$$\underset{s}{\operatorname{hlog}_x} z = \underset{s}{\operatorname{hlog}_x}(\underset{s-1}{\operatorname{hlog}_x} z) + 1 .$$

6 – A Connection to the Lambert's Function

The Lambert's Function is defined as the inverse function of the product-exponential function:

(20) $$x = y \cdot e^y$$

We shall call Product-Log its inverse function and indicate it as follows:

(21) $$y = \operatorname{plog}(x) = w(x)$$ [5]

Let us put in (20): $e^y = \sqrt[s]{e}$ square super-root of number *e*

which means: $^2(e^y) = (e^y)\,\hat{}\,(e^y) = e$

and: $e^{y \cdot e^y} = e$

therefore: $y \cdot e^y = 1$

The solution is: $y = \operatorname{plog}(1) = w(1) = \omega_1$ the Lambert's omega-one constant

with: $y = \omega_1 = 0.567143290409782$

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[5] The name ProductLog is adopted in the "*Mathematica*" software package, see also the Lambert w(x) function in http://mathworld.wolfram.com/LambertW-Function.html

The properties of the Lambert's $w(x) = \mathrm{plog}(x)$ function (and of the ω_1 constant) allow us to precisely calculate the square super-root of number e, as follows:

(22) $$\boxed{\overline{)e} = 1/\omega_1 = e^{\omega_1} = 1.7632228343519}$$

In a similar way, if we put: $a^y = \overline{)a}$

we get: $a^{y \cdot a^y} = a$

which means: $y \cdot a^y = y \cdot e^{y \cdot \ln a} = 1$

and, multiplying by ln a: $y \cdot \ln a \cdot e^{y \cdot \ln a} = \ln a$

therefore, we have: $y \cdot \ln a = \mathrm{plog}(\ln a)$

and, also: $y = \mathrm{plog}(\ln a) / \ln a$

which means: $\overline{)a} = a \wedge \left[\mathrm{plog}(\ln a)/\ln a\right]$

and which gives:

(23) $$\boxed{\overline{)a} = \ln a / \mathrm{plog}(\ln a)}$$

and which might also be simplified into:

(24) $$\overline{)a} = e \wedge \mathrm{plog}(\ln a)$$

Moreover, from formula (15), we know that: $\mathrm{sln}\, y = \mathrm{sln}(\ln y) + 1$. Then, by putting $y = \overline{)e}$,

we get: $\mathrm{sln}\left(\overline{)e}\right) = \mathrm{sln}\left(\ln\left(\overline{)e}\right)\right) + 1$.

But, remembering (22): $\ln\left(\overline{)e}\right) = 1/\overline{)e}$

we finally obtain:

(25) $$\boxed{\mathrm{sln}\left(\overline{)e}\right) - \mathrm{sln}\left(:\overline{)e}\right) = 1}$$

with: $:\overline{)e} = 1/\overline{)e}$

Finally, it is also interesting to notice that we can get three important generalizations of formulas (22), (24) and (25), taking into account that. In fact, from (24) we have:

$$\ln \overline{)a} = \ln\left(e \wedge \mathrm{plog}(\ln a)\right)$$

i.e: $\ln \overline{)a} = \mathrm{plog}(\ln a)$

and: $\log_a \overline{)a} = \log_a \left(a \wedge \left(\mathrm{plog}(\ln a)/\ln a\right)\right)$

i.e: $\log_a \overline{)a} = \mathrm{plog}(\ln a)/\ln a = 1/\overline{)a}$

Therefore, we have:

(26) $$\boxed{\ln \overline{)a} = \mathrm{plog}(\ln a) \quad \text{and} \quad \log_a \overline{)a} = 1/\overline{)a}}$$

and aslo

(27) $$\boxed{\mathrm{slog}_a\left(\overline{)a}\right) - \mathrm{slog}_a\left(:\overline{)a}\right) = 1}$$

7 – <u>The Infinite Tower</u>

From formula (12): $\quad {}^{2}\overline{)x} = 1/\,{}^{\infty}(1/x)$

we know that: $\quad {}^{\infty}(1/x) = 1/\,{}^{2}\overline{)x}$.

This expression allows us to find the expression for the infinite tower (infinite tetration) of x:

$$ {}^{\infty}(x) = 1/\,{}^{2}\overline{)1/x} = \text{plog}\left[\ln(1/x)\right]/\ln(1/x) $$

or, better:

(28) $$\boxed{{}^{\infty}(x) = \text{plog}(-\ln x)/(-\ln x)}$$

We also wish to show now that the infinite tower $y = {}^{\infty}(x)$, infinite tetration, is the inverse function of $x = \sqrt[y]{y}$. This can immediately be demonstrated as follows. In fact:

(29) $$\boxed{y = {}^{\infty}(x) \qquad \rightarrow \qquad x = {}^{\infty}\overline{)y} = \sqrt[y]{y}}$$ **QED**

8 – <u>The Tower Extension</u>

In ref. [1], paragraph 7.2 (*Incomplete Towers*) we mentioned an expression that we called an "*incomplete tower operation*", <u>incomplete</u> because the "last" exponent of the iterated exponentiation is different from the base. In particular, we introduced the following "exponentiation" operator:

(30) $\quad \boxed{a\,\wedge}\, p = a \wedge p$

and: $\quad \boxed{a\,\wedge}^{\,n}\, p = a \wedge (a \wedge (a \wedge (a \wedge \ldots.. p)))$ $\quad$ with n iterations of $a\,\wedge$

We have seen that any real number $z > 0$ can be expressed in the following way (the tetration canonical form), by means of an incomplete tower, as follows:

(31) $\quad z = \boxed{a\,\wedge}^{\,n} p = {}^{m}a = {}^{(n+q)}a \qquad 0 < q < 1, \quad 1 < p < a$

with:

a	base of the tower (tetration)
$m = n + q$	tower super-exponent
n	integer part of the super-exponent
q	super-mantissa
p	super-exponent extension (or tower extension)

By iteratively applying the "log" operator, we have also seen that it must be :

(32) $\quad p = {}^{q}a \qquad \leftrightarrow \qquad q = \text{slog}_a\, p$

For $q = 0$ we have $p = 1$ and for $q = 1$, $p = a$. By iteratively applying the "log" operator, for any z, it is always possible to determine the appropriate value of p. The determination of the non-integer super-exponent m is still a major problem, linked to the evaluation of the super-logarithm of p, partially solvable via appropriate approximations. Nevertheless, it is always possible to represent any real number z, by means of an incomplete tower, with base a, integer super-exponent n and super-exponent extension p, by means of a new operator "$*$", as follows (tetrational notation):

(33) $\quad \boxed{z = p * {}^{n}a} \qquad$ tetrational notation

Operator [6] "$*$" means that p (with $1 < p < a$) is the value of the super-exponent extension of ${}^{n}a$, according to the definition stipulated by formulas (30) and (31). Formula (33) is always computable.

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[6] Operator "$*$" is a kind of "gluing operator", appending the super-exponent extension in the last right position of the exponents' tower, as shown in (30), second formula.

9 – The Number Notation Hyper-format

In Mathematics and Physics, the most natural number notation is the traditional decimal fixed-point notation. However, few other important number notations are also used, according to the computational needs. For instance, the cyclotomic notation of number *x*, with base (modulo) *b*, can be shown as follows (*p* is called the value of *x*, modulo *b*, sometimes written as $p = x_{<b>}$)[7]:

(34) $$\boxed{x = p + (b \times n) = b \times \left(p/b + n \right)}$$ with: $0 \le p < b$, n natural

Another very important number notation is the classical scientific notation (also called the "floating point notation"), used in Physics and in Engineering, according to which a real number *x* can be represented as follows:

(35) $$\boxed{x = p \times (b \wedge n) = b \wedge \left(\log_b p + n \right)}$$ with: $1 \le p < b$, n natural

Quantity *b* is called the *base* of the notation (normally *b* = 10) and, as we know, *n* is the *order of magnitude* (base *b*) of *x*, while *p* is its *significant value (the significant)*.

We should like to propose here a new *tetrational notation of numbers*, using the *power-tower*, or *tower*, or *tetration* operation, according to formula (33).

(36) $$\boxed{x = p * (b\#n) = b\#\left(\mathrm{s}\log_b p + n \right)}$$ with: $1 \le p < b$, n natural

Number *b* is the "tower base", *n* is the integer super-exponent (or tower level) and *p* is the "*tower (super-exponent) extension*", that we can also write as follows:

(37) $$x = p * {}^{n}b\,.$$

In this new proposed notation of numbers, number *p* is the "*tower extension*" and the "*significant value*" of *x*, while *n* is the tetrational "*order of magnitude*". This notation is appropriate to represent very large numbers. For example, the Earth-Moon distance can be represented, by using both the floating point (power-of-ten) and the tetradic (tower-of-two) notations, in the following ways:

$$\mathbf{d}_{\mathrm{EM}} = 3.84401... \times (10^{8})\ \mathbf{m} \quad = 1.184698350... * ({}^{4}2)\ \mathbf{m}$$

i.e. two-tower-four, with the extension of 1.18469835… (etc.). We could also say that 4 is the "tetrational order of magnitude, base 2, of number $|\mathbf{d}_{\mathrm{EM}}|$". In case of very large numbers, the tetrational notation appears very useful. For instance we could, very compactly, write:

$$x = 10^{\,7.2742\ \mathrm{x}\ 10\,\wedge\,40} = 1.5 * ({}^{5}2).$$

In conclusion, we may define a generalized hyper-format for number notation, based on three numerical components (*p, b, n*), interlinked by two operators (generally indicated as $\oplus, \otimes$) for representing and noting a real number *x*, as follows [8]:

(38) $$\boxed{x = p \oplus \left(b \otimes n \right)}$$

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[7] Modular arithmetic, also called "clock arithmetic".

[8] RRH (Rubtsov-Romerio Hyper-format), © C. A. Rubtsov and G. F. Romerio 2003, 2010, SIAE deposit (Rome) no. 0501678 of 21-04-2005.

The three numerical components are: p (hyper-exponent extension, the significant value), b (base of the notation), n (order of magnitude, also called number of cycles in case of the cyclotomic notation). The meaning of the ($\oplus, \otimes$) operators is given in the following table:

	General operator $\oplus$	General operator $\otimes$
Cyclotomic notation	+ (addition)	× (multiplication)
Scientific floating point notation	× (multiplication)	^ (exponentiation)
Tetradic notation	∗ (tower extension)	# (tetration, iterated exponentiation)

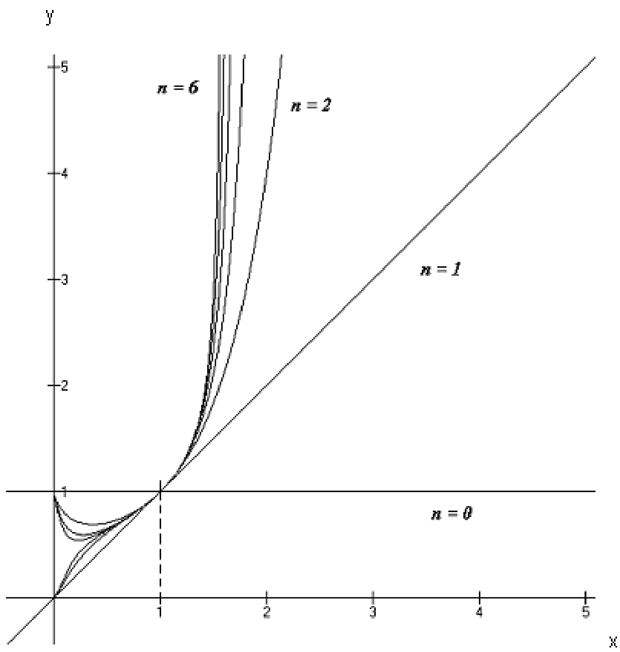
The hyper-format for number notation, so defined, takes into account the fact that all the operators of the $\otimes$ series belong to the infinite family of the hyper-operators (°, +, ×, ^, #, …), each one corresponding to the iteration of that of the lower level, and those of the $\oplus$ series are the corresponding operators indicating the appropriate hyper-exponent extension of the "last" iteration with number p.

10 – Tower Plots

Let us start from the following expression of a tetration operation, from (1), for $s = 4$:

(39) $$y = {}^{z}x$$ x-tower-z

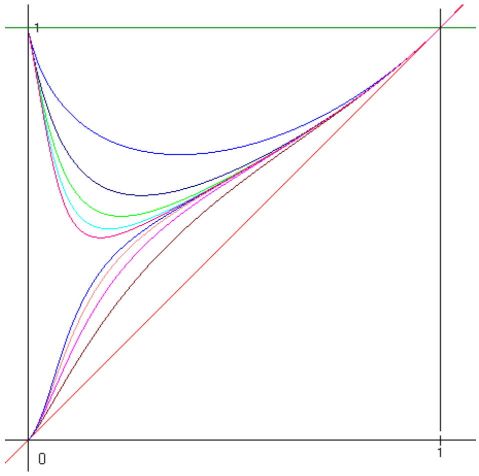
A set of plots of tower functions [9] can be easily obtained of this relation, for $z = n$ (with $n \in \mathbb{N}$), as shown in the following figure taken from [1], paragraph 5.5 (the tower functions with super-exponent n, or n super-degree). In the following example n = 0, 1, 2, 3, 4, 5, 6.



A lot of research has been carried on, since the Euler's times, concerning the domain $0 < x < 1$, where a peculiar phenomenon suddenly appears in the region of about $0 < x < 0.06$. Euler himself, in fact, found a kind of "off-limits" zone, where the $y = {}^{n}x$ curves, for n integer, were not allowed. By increasing n, i.e. for $n \to \infty$, the plots are obliged to approach a line, functioning like a kind of "attractor", without trespassing it.

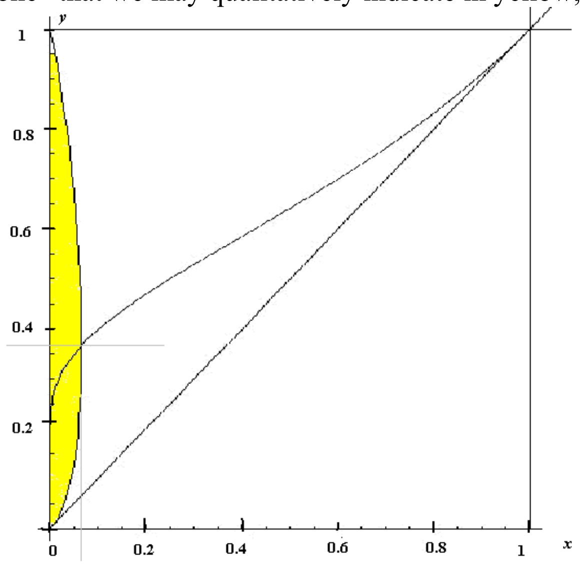
[9] The tower functions are a set of functions defined for z constant, in our case for $z = n$, with n natural.

To give a first idea of that, let us expand the part of the diagrams in the $0 < x < 1$ domain.



The curves with an even n (here: $n = 2, 4, 6, 8, 10$) are in the upper part of the diagram, passing by point (0, 1), those with an odd n (n = 1, 3, 5, 7, 9) are in the lower part, all passing by point (0,0).

This area has also been recently analysed by I. N. Galidakis, professor at the University of Krete, who published in the Internet a study showing a family of curves, up to $n = 30$. [10] It appears what we might really call the "off-limits zone" that we may qualitatively indicate in yellow, as follows:



The coordinates of a first vertex point (VP1, the Hopf's bifurcation point) as shown in the plot, i.e. the maximum value of x in the yellow area, are:

(40) $$x = e^{-e} \qquad y = e^{-1}$$

All the tower plots, for $n \to \infty$, are attracted both by the curve representing the "infinite tower" and by the perimeter of the "off-limits zone". As we know, he attractor line is defined by formula (26):

$$y = {}^{\infty}(x) = \text{plog}(-\ln x)/(-\ln x)$$

the plot of which is shown in [1], Appendix A-06 (see next page).

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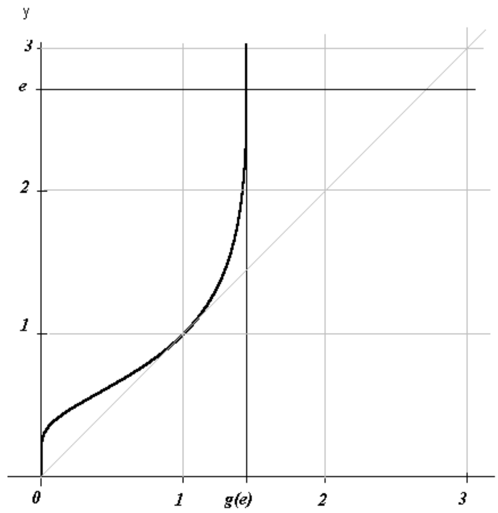
[10] See also: http://users.forthnet.gr/ath/jgal/math/hyperpower.html.



Coordinates: $x = \mathrm{g}(e) = e^{1/e}$
$y = e$

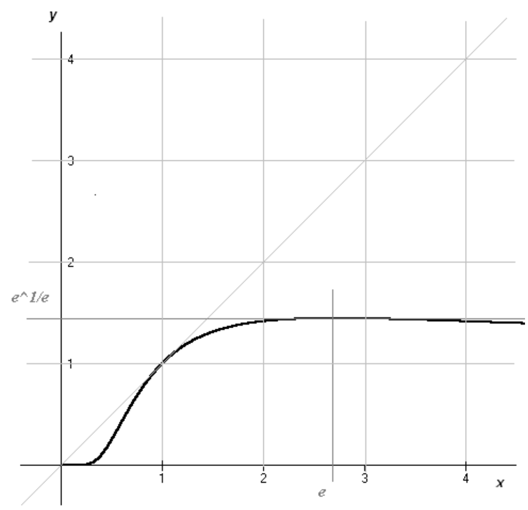
indicate another vertex point, VP2, such that, for $x > e^{1/e}$, the infinite tower in *unlimited* (infinite).

The equation of the perimeter of the off-limits zone remains to be precisely defined.
The coordinates of the two vertex points are as follows:

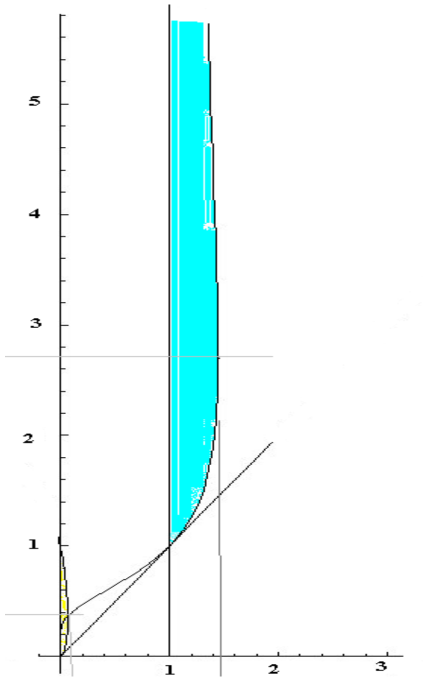
(41) VP1 → $x = e^{-e} =$ 0. 065988036
$y = e^{-1} =$ 0. 367879441

VP2 → $x = e^{1/e} =$ 1. 444667861
$y = e =$ 2. 718281828

The coordinates of the two points can be transformed, one into the other, by operating the following exchange: $e \rightarrow 1/e$. In fact, we know that function $y = \mathrm{plog}(-\ln x)/(-\ln x)$ is itself the inverse function of $y = \sqrt[x]{x}$, the plot of which is as follows (see [1] Fig A1, Annex A-06):



This curve has a maximum for $x = e$, with $y = e^{1/e}$, and a horizontal asymptote at $y = 1$. Its complete inverse function should appear as follows:



11 – Tetrational Approximations

Let us consider again the following expression:

(42) $$y = {}^{n}b \qquad n \in N, n > 0 \quad \text{and} \quad b > 0$$

Let us start recalling that, in the scientific literature, it is commonly stated that "the infinite tower $y = {}^{\infty}b$ is only converging for values of the base *b* such that:

(43) $$0.065988036 = e^{-e} \leq b \leq e^{1/e} = 1.444667861\text{''}.$$

Actually, to be more precise and with reference to the plots of the last figure, we should define four domains defined by the values assumed by the base *b*:
(44)

- *I.* $0 \leq b < e^{-e}$, where $y = {}^{\infty}b$ <u>does not converge</u> and it is *undetermined*, yellow area;
- *II.* $e^{-e} \leq b < 1$, where $y = {}^{\infty}b$ <u>converges</u> to $y = \text{plog}(-\ln b)/(-\ln b)$;
- *III.* $1 < b \leq e^{1/e}$, where $y = {}^{\infty}b$ <u>converges</u> and it is a two branches function, the lower branch being described by $y = \text{plog}(-\ln b)/(-\ln b)$, blue area;
- *IV.* $e^{1/e} < b < \infty$, I where $y = {}^{\infty}b$ <u>diverges</u> because it is *unlimited*.

Let us now examine expression (42) in which base *b* is constant and *n* is variable. We obtain what we might call a tetrational function, similar (at a higher hierarchical level) to the exponential function. In case possibility of extension of tetration to real super-exponents, we should have:

(45)

$$y = {}^{z}b \qquad \text{tetrational function}$$

$$y = b^{z} \qquad \text{exponential function}$$

From (30), (36) and (37), we know that we can write a tetrational function as follows:

(42) $$\boxed{y = {}^{z}b = p * {}^{n}b = {}^{n+q}b}$$

with:

z	real super-exponent	
n	integer part of z	
q	fractional part of z, super-mantissa, $0 \le z < 1$	
p	tower extension, $1 \le p < b$	

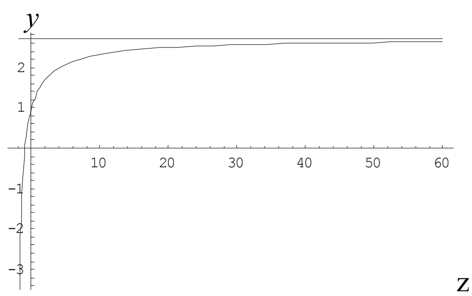
and we also have:

$$\boxed{q = \text{slog}_b\, p} \quad \text{and} \quad \boxed{p = {}^{q}b}$$

As we know, the super-exponent or tower extension p is easily computable, by applying iterated logarithms, base b, to y. The calculation of super-mantissa q is possible if we know how to evaluate the super-logarithm of a number. The solution of this very important problem will allow the full extension of tetration to the real numbers. This problem, until now, was not yet solved. Nevertheless, a very interesting approximation is possible, for values of the base $b = \sim e$.

The various values asymptotically assumed by $y = {}^{\infty}b$, in the four regions described in (44), indicate that the behaviour of $y = {}^{z}b$ can be very different in those regions. In particular, for $b = 1$, we have a singular situation in which we must have $y = 1$ (constant).

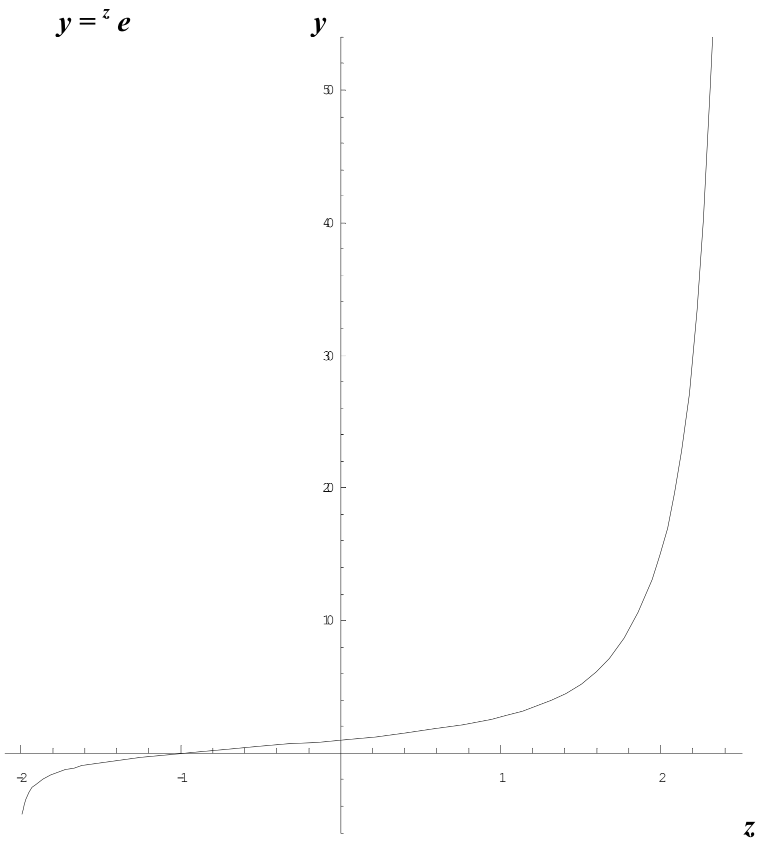
Another important critical plot of $y = {}^{z}b$ is detectable for $b = e^{1/e} = 1.\ 444667861..$, which should give qualitatively something like the plot shown in the following figure (only natural values of z are satisfying expression $y = {}^{z}b$, the continuous graph, y against z, is only a qualitative extension):



In fact, the hypothetical continuous graph of $y = {}^{z}(\sqrt[e]{e})$ should show a vertical asymptote for $z = -2$ and a horizontal asymptote at $y = e$, for $z \to \infty$. An interesting approximation of $y = {}^{z}b$, for $z = e$, can be built-up by supposing to identify the two following expressions:

$${}^{z}e =\sim e^{z} \qquad \text{for } 1 \le z < 2$$

This is justified by the fact that, for $b = e$, with this assumption, we obtain a smooth and regular plot, satisfying all the points of expression $y = {}^{z}b$ (for $z = n$, natural), with a vertical asymptote for $z = -2$ and without sensible tangent discontinuities, as follows:



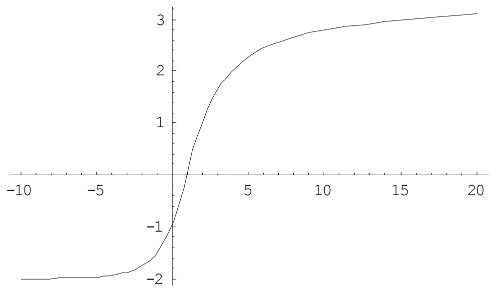
The last graph represents the key function:

$y = {}^{z}e$ the natural tower (base ***e***)

Its inverse function is the super logarithm (base ***e***) of ***z***.

$z = \text{slog}_{e}(y) = \text{sln } y$ the natural superlog (base *e*).

The plot of a similar function, i.e. of $z = \text{slog}_2 (y)$, the super-logarithm of *y*, base two, which might assume a great importance for the representation of very large numbers, is as follows:



It appears as a regular, always increasing curve, with a horizontal asymptote at $z = -2$.

Bibliography

[1] - Constantin A. Rubtsov, Giovanni F. Romerio - Ackermann's Function and New Arithmetical Operations
Paper cited in the NKS Bibliography, see also: http://www.rotarysaluzzo.it/filePDF...zioni%20(1).pdf